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A Frame Bundle Generalization of Multisymplectic Momentum Mappings ^{*†}

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Abstract

We construct momentum mappings for covariant Hamiltonian field theories using a generalization of symplectic geometry to the bundle $L_V Y$ of vertically adapted linear frames over the bundle of field configurations Y . Field momentum observables are vector-valued momentum mappings generated from automorphisms of Y , using the $(n + k)$ -symplectic geometry of $L_V Y$. These momentum observables on $L_V Y$ generalize those in covariant multisymplectic geometry and produce conserved field quantities along flows. Three examples illustrate the utility of these momentum mappings: orthogonal symmetry of a Kaluza-Klein theory generates the conservation of field angular momentum, affine reparametrization symmetry in time-evolution mechanics produces a version of the parallel axis theorem of rotational dynamics, and time reparametrization symmetry in time-evolution mechanics gives us an improvement upon a parallel transport law.

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1 Introduction

Norris' generalization [15, 16, 17] of the symplectic geometry of the cotangent bundle has been a successful theoretical tool for particle mechanics. Norris has shown that, for an n -dimensional manifold M , the standard symplectic geometry of the cotangent bundle T^*M may be obtained entirely from the *n -symplectic* geometry of the linear frame bundle LM . The key component of n -symplectic geometry is recognizing that the canonical soldering one-form [9] on LM may be employed as a vector-valued n -symplectic potential.

Subsequently, the author, together with Fulp and Norris, has shown that the *multisymplectic* geometry of the affine multiphase space Z introduced by Kijowski [7, 8] and refined by Gotay, et al. [6], may be generalized by adapting Norris's theory to $L_V Y$, the bundle of vertically adapted linear frames of a configuration bundle Y of a classical field [4, 10]. The new theory reproduces the Poisson bracket of momentum observables defined on Z . However, the set of momentum observables on Z is not closed under this bracket, whereas the analogous set of momentum observables on $L_V Y$ is closed under their Poisson bracket [4, 10].

The purpose of this paper is to develop $(n+k)$ -symplectic momentum mappings on $L_V Y$ (where n is the dimension of the parameter space and k the dimension of the range of the field), and to use examples to illustrate some advantages of the $(n+k)$ -symplectic approach to classical field theories. In Hamiltonian classical mechanics, momentum mappings are the foundation for obtaining conserved quantities from Lie group symmetries of phase space (the symplectic manifold), thus linking the group-theoretic aspects of a mechanical system to the canonical aspects. In applications to classical field theories, the momentum mapping in [6] is spacetime covariant, connecting Lie group symmetries to the canonical structures of the covariant field theory. We show that the generalized symplectic structure of $L_V Y$ is invariant under automorphisms of Y lifted to $L_V Y$ and that momentum mappings on $L_V Y$ produce the momentum mappings on Z found in [6]. This is analogous to the work of Norris [16], in which the momentum mappings of standard symplectic geometry on T^*M are generated from n -symplectic geometry on LM . A momentum observable on $L_V Y$ is a special case of a momentum mapping obtained from the Lie algebra of projectable vector fields on Y .

As an example, we consider the $(n+k)$ -symplectic momentum mapping constructed from the infinitesimal generator of an orthogonal group. We obtain conserved quantities along flows that give us frame bundle versions of conservation of field angular momentum in a Kaluza-Klein theory. In time-evolution mechanics we develop two examples: Euclidean group symmetry leads to a frame bundle version of a "parallel axis theorem," and time-reparametrization symmetry enables us to extend a result of Norris [15, 17] regarding parallel transport of frames along geodesics of a Riemannian metric.

The format of this paper is as follows. We review momentum mappings, first in multisymplectic geometry in Section 2, and then in n -symplectic geometry in Section 3. After a brief summary of the $(n + k)$ -symplectic geometry of $L_V Y$ in Section 4, we introduce momentum mappings on $L_V Y$ in Section 5. In Section 6 we produce momentum mappings on Z from those on $L_V Y$, and in Section 7 we derive conserved quantities. Examples are found in Section 8.

2 Momentum mappings in multisymplectic geometry

We review the multisymplectic geometry of Gotay, et al. [6] and momentum mappings in this context. Let X be an oriented n -dimensional manifold and let $\pi_{XY} : Y \rightarrow X$ be a fiber bundle with standard fiber a k -dimensional manifold. (Note: In general, we shall denote a projection from A onto B as π_{BA} .) A classical field is a section of the **field configuration space** Y over the parameter space X . From local coordinates $\{x^i\}$, $i = 1, \dots, n$ on X we may construct local adapted coordinates $\{x^i, y^A\}$, $i = 1, \dots, n$, $A = 1, \dots, k$, on Y . The **multivelocit**y bundle is the first-order jet bundle JY , the affine bundle over Y whose fiber over $y \in Y$ consists of linear maps $\gamma_y : T_{\pi_{XY}(y)}X \rightarrow T_y Y$ satisfying $\pi_{XY*} \circ \gamma_y = \text{Id}_{T_{\pi_{XY}(y)}X}$. The **bundle of affine cojets** [5, 6] J^*Y is the vector bundle over Y whose fiber at y is the set of affine maps from $J_y Y$ to $\wedge^n_{\pi_{XY}(y)} X$. It follows that $\dim J^*Y = \dim JY + 1$.

An equivalent description [6] of J^*Y is useful. Define the vertical subbundle of TY to be the fiber bundle $V(TY)$ whose fiber over $y \in Y$ is

$$V(T_y Y) := \{w_y \mid y \in Y, w_y \in T_y Y \text{ and } \pi_{XY*}(w_y) = 0\}.$$

The **multiphase space** Z [6, 8] is the fiber bundle whose fiber Z_y is

$$Z_y := \{z \in \wedge^n Y \mid v \lrcorner w \lrcorner z = 0 \ \forall v, w \in V(T_y Y)\},$$

where \lrcorner denotes the inner product of a vector with a differential form. The bundle Z , originally defined by Kijowski [8], admits a canonical n -form,

$$\Theta(z) = \pi_Y^*(z),$$

which is the pullback via inclusion $Z \hookrightarrow \wedge^n Y$ of the canonical n -form on $\wedge^n Y$. We can define coordinates $\{x^i, y^A, p_B^j, p\}$ on Z where $\{x^i, y^A\}$ are the lifts of the adapted coordinates of Y ,

$$\begin{aligned} p(z) &= \frac{\partial}{\partial x^n} \lrcorner \cdots \lrcorner \frac{\partial}{\partial x^1} \lrcorner z, \quad \text{and} \\ p_B^j(z) &= (-1)^{j-1} \frac{\partial}{\partial x^n} \lrcorner \frac{\partial}{\partial x^{n-1}} \lrcorner \cdots \lrcorner \widehat{\frac{\partial}{\partial x^j}} \lrcorner \cdots \lrcorner \frac{\partial}{\partial x^1} \lrcorner \frac{\partial}{\partial y^B} \lrcorner z, \end{aligned} \tag{2.1}$$

where $\widehat{\frac{\partial}{\partial x^j}}$ denotes the omission of $\frac{\partial}{\partial x^j}$. If we define in local lifted coordinates

$$d^n x := dx^1 \wedge \cdots \wedge dx^n, \quad d^{n-1} x_i := \frac{\partial}{\partial x^i} \lrcorner d^n x, \quad \text{and} \quad d^{n-2} x_{ij} := \frac{\partial}{\partial x^j} \lrcorner \frac{\partial}{\partial x^i} \lrcorner d^n x,$$

then Θ may be expressed locally as

$$\Theta = p_A^i dy^A \wedge d^{n-1}x_i + pd^nx$$

The nondegenerate $(n+1)$ -form $d\Theta$ is considered a ***multisymplectic structure*** form on Z . The pair $(Z, d\Theta)$ is called a ***multisymplectic manifold*** [6].

Definition [5, 6] Let a Lie group \mathcal{G} act on the left on Z and the multisymplectic structure form $d\Theta$ be invariant under this action. Let \mathfrak{g} be the Lie algebra of \mathcal{G} . A mapping

$$J : Z \rightarrow \mathfrak{g}^* \otimes \wedge^{n-1} Z$$

is a ***momentum mapping*** if J covers the identity on Z and \hat{J} , defined by

$$\hat{J}(\xi)(z) = \langle J(z), \xi \rangle ,$$

satisfies

$$d\hat{J}(\xi) = -\xi_Z \lrcorner d\Theta ,$$

where ξ_Z is the infinitesimal generator of the \mathcal{G} -action on Z induced by $\xi \in \mathfrak{g}$.

A ***momentum observable*** on Z is an $(n-1)$ -form f on Z that satisfies

$$df = -X \lrcorner d\Theta \tag{2.2}$$

for some vector field X on Z . The vector field X is called a ***Hamiltonian vector field*** on Z . By definition, $\hat{J}(\xi)$ is a momentum observable and ξ_Z is the corresponding Hamiltonian vector field.

The group $\text{Aut } Y$ of fiber bundle automorphisms of Y over X in the multisymplectic geometry of Z is the analogue of the group $\text{Diff } X$ of diffeomorphisms of X in the symplectic geometry of T^*X . Let $\mathcal{X}Y$ be the Lie algebra of vector fields on Y . Denote the vector space of complete vector fields of Y that are projectable to X by $\mathcal{X}_{\text{Proj}}Y$. Note that $\mathcal{X}_{\text{Proj}}Y$ is a Lie subalgebra of $\mathcal{X}Y$, since $[\pi_{XY*}v, \pi_{XY*}w] = \pi_{XY*}[v, w]$. Formally, $\mathcal{X}_{\text{Proj}}Y$ is the Lie algebra of $\text{Aut } Y$. The proof is a routine extension of the proof in [1] that $\mathcal{X}X$ is the formal Lie algebra of $\text{Diff } X$, provided that we identify $\mathcal{X}Y$ with the tangent space at the identity of $\text{Diff } Y$ and $\mathcal{X}_{\text{Proj}}Y$ with the tangent space at the identity of $\text{Aut } Y$. However this ignores the topology of Y . Milnor [13] notes that if Y is compact then $\text{Diff } Y$ is a Lie group that admits a C^∞ manifold structure. For noncompact Y we may choose only to consider diffeomorphisms which are the identity outside a compact subset of Y . This leads to vector fields that vanish outside this subset. There is also a slight ambiguity because $\mathcal{X}_{\text{Proj}}Y$ serves both as the Lie algebra and as the collection of infinitesimal generators. To resolve this, let $[\xi, \zeta]$ denote the Lie bracket defined on the formal Lie algebra of left invariant vector fields on the manifold $\text{Aut } Y$, and let $[\xi_Y, \zeta_Y]$ denote the usual Lie bracket defined on $\mathcal{X}Y$ (in which the infinitesimal generators of the action of $\text{Aut } Y$ on Y are right invariant). See [1, Exercise 4.1G] for a clarification.

The ***canonical lift*** [6] of $\eta_Y \in \text{Aut } Y$ is a map

$$\eta_Z : Z \rightarrow Z : z \mapsto (\eta_Y^{-1})^*(z) .$$

The map η_Z is a π_{XZ} -bundle map under which Θ is invariant. For $\xi \in \mathcal{X}_{\text{Proj}} Y$, it follows that $\mathbf{L}_{\xi_Z} \Theta = 0$ and the induced momentum mapping defines a momentum observable

$$\hat{J}(\xi)(z) := \xi_Z \lrcorner \Theta(z) = \pi_{YZ}^*(\xi_Y \lrcorner z). \quad (2.3)$$

It follows that \hat{J} is Ad^* equivariant, that is, $J(\text{Ad}_\eta^{-1} \xi) = \eta_Z^*(\hat{J}(\xi))$. Let $T^1(Z)$ denote the vector space of momentum observables, and observe that $\xi \mapsto \hat{J}(\xi)$ is a bijection from $\mathcal{X}_{\text{Proj}} Y$ to $T^1(Z)$.

If in local adapted coordinates we write $\xi \in \mathcal{X}_{\text{Proj}} Y$ as $\xi = \xi^i(x^j) \frac{\partial}{\partial x^i} + \xi^A(x^j, y^B) \frac{\partial}{\partial y^A}$, then

$$\hat{J}(\xi)(z) = (p_A^i \xi^A + p \xi^i) d^{n-1} x_i - p_A^i \xi^j dy^A \wedge d^{n-2} x_{ij} \quad (2.4)$$

and

$$\xi_Z = \xi^k \frac{\partial}{\partial x^k} + \xi^A \frac{\partial}{\partial y^A} + \left(p_A^j \frac{\partial \xi^i}{\partial x^j} - p_A^i \frac{\partial \xi^j}{\partial x^j} - p_B^i \frac{\partial \xi^B}{\partial y^A} \right) \frac{\partial}{\partial p_A^i} - \left(p \frac{\partial \xi^i}{\partial x^i} + p_A^i \frac{\partial \xi^A}{\partial x^i} \right) \frac{\partial}{\partial p}. \quad (2.5)$$

If we define a Poisson bracket by

$$\{\hat{J}(\xi), \hat{J}(\zeta)\} := -d\Theta(\xi_Z, \zeta_Z) \quad (2.6)$$

then using $\mathbf{L}_{\xi_Z} \Theta = 0$ and the Lie derivative identities [1, p. 121],

$$[X, Y] \lrcorner \alpha = \mathbf{L}_X(Y \lrcorner \alpha) - Y \lrcorner (\mathbf{L}_X \alpha) \quad \text{and} \quad \mathbf{L}_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha), \quad (2.7)$$

we obtain

$$\{\hat{J}(\xi), \hat{J}(\zeta)\} = \hat{J}([\xi, \zeta]) - d(\xi_Z \lrcorner \zeta_Z \lrcorner \Theta). \quad (2.8)$$

For an alternative proof using the Ad^* equivariance of J , see [6, Prop. 4.5].

The Poisson bracket on $T^1(Z)$ given by (2.6) is not a true Poisson bracket because there lacks an associative multiplication of $(n-1)$ -forms on which the bracket acts as a derivation. Worse, $d(\xi_Z \lrcorner \zeta_Z \lrcorner \Theta)$ is not in $T^1(Z)$ because from equation (2.2) the Hamiltonian vector field of an exact form is the zero vector field on Z , but the momentum observable corresponding to the zero vector field is the zero $(n-1)$ -form on Z . Thus, $T^1(Z)$ is **not closed under the Poisson bracket**. Equation (2.8) explains the remark in [6] that the Poisson bracket of two momentum observables “is up to the addition of exact terms, another momentum observable.”

3 Momentum mappings in n -symplectic geometry

Here we briefly review Norris’s program of n -symplectic geometry [15, 16, 17, 18]. For an n -dimensional manifold M , the linear frame bundle LM is the $GL(n, \mathbb{R})$ principal fiber bundle

$$\{(m, \{e_i\}) \mid m \in M, \{e_i\}, i = 1, 2, \dots, n \text{ is a basis for } T_m M\}.$$

We shall regard the \mathbb{R}^n -valued canonical soldering one-form θ (see [9]) as an n -symplectic potential, and $d\theta$, a closed nondegenerate two-form, serves as an \mathbb{R}^n -valued **n -symplectic structure** form. If $u = (e, \{e_i\}) \in LM$, $\lambda : LM \rightarrow M$ is the canonical projection, and $\{r_i\}$

is the standard basis for \mathbb{R}^n , then $\theta_u(X) = e^i(d_u\lambda(X))r_i = \theta_u^i(X)r_i$, or in local canonical coordinates, $\theta^i = \pi_j^i dx^j$.

The structure equation for n -symplectic geometry may take the general symmetrized form

$$d\hat{f}^{(i_1 i_2 \dots i_p)} = -p! X_{\hat{f}}^{(i_1 i_2 \dots i_{p-1}} \lrcorner d\theta^{i_p)}. \quad (3.1)$$

Here $\hat{f} = (\hat{f}^{(i_1 i_2 \dots i_p)}) : LM \rightarrow \otimes_s^p \mathbb{R}^n$ (where \otimes_s denotes the symmetric tensor product) is a **symmetric Hamiltonian observable**, and $X_{\hat{f}} = (X_{\hat{f}}^{i_1 i_2 \dots i_{p-1}})$ is the corresponding \mathbb{R}^n -valued **Hamiltonian vector field** (for $1 \leq i_\alpha \leq n$, $1 \leq \alpha \leq p$). Equation (3.1) may be also be expressed in antisymmetric form, employing antisymmetric Hamiltonian observables, $(\hat{f}^{[i_1 i_2 \dots i_p]}) : LM \rightarrow \wedge^p \mathbb{R}^n$. The complete set of symmetric Hamiltonian observables (for $p \geq 1$) admits a naturally defined Poisson bracket, under which the set forms a Poisson algebra, while the complete set of antisymmetric Hamiltonian observables forms a graded Poisson algebra with respect to its natural bracket. These brackets are the frame bundle versions of the Schouten-Nijenhuis brackets [18].

Definition [16] Let \mathcal{G} be a Lie group with an action on LM under which $d\theta$ is invariant. Let \mathfrak{g} be the Lie algebra of \mathcal{G} . A mapping $J : LM \rightarrow \mathfrak{g}^* \otimes \mathbb{R}^n$ is a **momentum mapping** if

$$d\hat{J}(\xi) = -\xi_{LM} \lrcorner d\theta \quad \forall \xi \in \mathfrak{g}$$

where ξ_{LM} is the infinitesimal generator of the \mathcal{G} -action on LM generated by $\xi \in \mathfrak{g}$ and $\hat{J}(\xi) : L_V Y \rightarrow \mathbb{R}^n$ is defined by

$$\hat{J}(\xi)(w) = \langle J(w), \xi \rangle \quad \forall w \in LM.$$

Thus, $\hat{J}(\xi)$ is a (symmetric/antisymmetric) Hamiltonian observable (for $p = 1$) and ξ_{LM} is its Hamiltonian vector field.

Norris has shown [16] that we can produce the symplectic structure of T^*M from the n -symplectic structure of LM and the vector space of polynomial observables on T^*M (a set dense in $C^\infty(T^*M)$) from the symmetric Hamiltonian observables on LM . He also has shown [16] that a momentum mapping associated to the lifted action of $\text{Diff } M$ on LM induces a momentum mapping on T^*M associated to the lifted action of $\text{Diff } M$ on T^*M . For further details the reader should consult the literature [3, 4, 15, 16, 17, 18].

4 $(n+k)$ -symplectic geometry on the vertically adapted frame bundle

For the $(n+k)$ -dimensional fiber bundle Y , the **vertically adapted frame bundle** is defined by

$$L_V Y := \{(y, \{e_i, \epsilon_A\}) \in LY \mid \{\epsilon_A\} \text{ is a frame of } V(T_y Y)\}.$$

We shall review $(n+k)$ -symplectic geometry on $L_V Y$. For a more detailed summary, see [4, 10].

The bundle $L_V Y$ is a reduced subbundle of LY obtained by breaking the $GL(n+k)$ symmetry of LY [10]. The structure group of $L_V Y$ is the matrix group

$$G_A := \left\{ (N, K, A) = \begin{pmatrix} N & 0 \\ A & K \end{pmatrix} \mid N \in GL(n), K \in GL(k), A \in \mathbb{R}^{k \times n} \right\},$$

with multiplication given by block matrix multiplication. The free right action of G_A on $L_V Y$ is given by

$$(y, \{e_i, \epsilon_A\}) \cdot (N, K, A) := (y, \{e_j N_i^j + \epsilon_B A_i^B, \epsilon_B K_A^B\}). \quad (4.1)$$

Let $(y, \{e^i, \epsilon^A\})$ be the coframe dual to the vertically adapted frame $(y, \{e_i, \epsilon_A\})$. We may define the right action of G_A on $(y, \{e^i, \epsilon^A\})$ by

$$(y, \{e^i, \epsilon^A\}) \cdot (N, K, A) = (y, \{(N^{-1})_j^i e^j, -(K^{-1} A N^{-1})_j^A e^j + (K^{-1})_B^A \epsilon^B\}). \quad (4.2)$$

The coframe in (4.2) is dual to the frame $(y, \{e_i, \epsilon_A\}) \cdot (N, K, A)$.

The pullback of the $(n+k)$ -symplectic structure form $d\theta$ via inclusion $i : L_V Y \hookrightarrow LY$ is closed and nondegenerate on $L_V Y$, just as $d\theta$ is on LY [4]. Local canonical coordinates on $L_V Y$ are $\{x^i, y^A, \pi_j^i, \pi_B^A, \pi_i^A\}$, where $\{x^i, y^A\}$ are local adapted coordinates on Y , $\pi_j^i := e^i(\frac{\partial}{\partial x^j})$, $\pi_B^A := \epsilon^A(\frac{\partial}{\partial y^B})$, and $\pi_j^A := \epsilon^A(\frac{\partial}{\partial x^j})$. Let $\{r_i\}_{i=1,2,\dots,n}$ be the standard basis of \mathbb{R}^n and $\{s_A\}_{A=1,2,\dots,k}$ be the standard basis of \mathbb{R}^k . Define $\hat{r}_i := (r_i, 0) \in \mathbb{R}^{n+k}$ and $\hat{s}_A := (0, s_A) \in \mathbb{R}^{n+k}$. Then, $\{R_\mu\}_{\mu=1,2,\dots,n+k} = \{\hat{r}_i, \hat{s}_A\}_{i=1,2,\dots,n; A=1,2,\dots,k}$ is the standard basis of \mathbb{R}^{n+k} . In local coordinates,

$$i^* \theta = \pi_j^i dx^j \hat{r}_i + (\pi_i^A dx^i + \pi_B^A dy^B) \hat{s}_A. \quad (4.3)$$

Define the vector space $HF^1(L_V Y)$ of Hamiltonian observables on $L_V Y$ to be the collection of functions $\hat{f} : L_V Y \rightarrow \mathbb{R}^n$ such that $d\hat{f} = -X_{\hat{f}} \lrcorner i^* d\theta$. The vector space of corresponding Hamiltonian vector fields $X_{\hat{f}}$ on $L_V Y$ is denoted by $HV^1(L_V Y)$. Finally, define $T^1(L_V Y)$ to be the vector space of tensorial \mathbb{R}^{n+k} -valued functions on $L_V Y$. An element of $T^1(L_V Y)$ can be expressed in local coordinates as

$$\hat{f} = f^i(x^k, y^C) \pi_i^j \otimes \hat{r}_j + \left(f^A(x^k, y^C) \pi_A^B + f^i(x^k, y^C) \pi_i^B \right) \otimes \hat{s}_B.$$

For $\hat{f} \in T^1(L_V Y)$ we compute $X_{\hat{f}}$ locally. This yields

$$X_{\hat{f}} = f^i \frac{\partial}{\partial x^i} + f^A \frac{\partial}{\partial y^A} - \frac{\partial f^k}{\partial x^j} \pi_k^i \frac{\partial}{\partial \pi_j^i} - \frac{\partial f^C}{\partial y^B} \pi_C^A \frac{\partial}{\partial \pi_B^A} - \left(\frac{\partial f^j}{\partial x^i} \pi_j^A + \frac{\partial f^B}{\partial x^i} \pi_B^A \right) \frac{\partial}{\partial \pi_i^A},$$

subject to the constraints on \hat{f} ,

$$\frac{\partial f^i}{\partial y^A} = 0 \quad \forall i = 1, \dots, n \text{ and } A = 1, \dots, k.$$

Thus, $T^1(L_V Y) \not\subset HF^1(L_V Y)$, so we shall define $T_V^1(L_V Y) := HF^1(L_V Y) \cap T^1(L_V Y)$. The following proposition is proven in [10].

Proposition 4.1 $T_V^1(L_V Y)$, $T^1(Z)$, and $\mathcal{X}_{\text{proj}} Y$ are in pairwise bijective correspondence. Furthermore, if $n \geq 2$ and $k \geq 2$ then $HF^1(L_V Y) \simeq T_V^1(L_V Y) \oplus C^\infty(X, \mathbb{R}^n) \oplus C^\infty(Y, \mathbb{R}^k)$.

5 Momentum mappings on the vertically adapted frame bundle

Definition Let \mathcal{G} be a Lie group with an action on $L_V Y$ under which $i^*d\theta$ is invariant. Let \mathfrak{g} be the Lie algebra of \mathcal{G} . A mapping $J : L_V Y \rightarrow \mathfrak{g}^* \otimes \mathbb{R}^{n+k}$ is a **momentum mapping** if

$$d\hat{J}(\xi) = -\xi_{L_V Y} \lrcorner i^*d\theta,$$

where $\xi_{L_V Y}$ is the infinitesimal generator of the \mathcal{G} -action on $L_V Y$ generated by $\xi \in \mathfrak{g}$ and $\hat{J}(\xi) : L_V Y \rightarrow \mathbb{R}^{n+k}$ is defined by

$$\hat{J}(\xi)(w) = \langle J(w), \xi \rangle.$$

Clearly, $\hat{J}(\xi) \in HF^1(L_V Y)$ and $\xi_{L_V Y} \in HV^1(L_V Y)$. Note that if $L_V Y$ is invariant under an action of \mathcal{G} on LY then the momentum mapping on LY (as defined in Section 3) restricts to a momentum mapping on $L_V Y$.

Now we consider momentum mappings when $\mathcal{G} = \text{Aut } Y$. Define $\text{Aut}(L_V Y)$ to be the subgroup of $\text{Diff}(L_V Y)$ whose elements are fiber bundle automorphisms both over Y and over X . Let $\eta_Y \in \text{Aut } Y$ cover $\eta_X \in \text{Diff } X$. Define the mapping

$$\eta_{L_V Y} : L_V Y \rightarrow L_V Y : (y, \{e_i, \epsilon_A\}) \mapsto (\eta_Y(y), \{\eta_{Y*}e_i, \eta_{Y*}\epsilon_A\}).$$

Clearly, $\eta_{L_V Y}$ is bijective and smooth and has a smooth inverse. By definition, $\pi_{Y|L_V Y} \circ \eta_{L_V Y} = \eta_Y \circ \pi_{Y|L_V Y}$. Since $\pi_{XY*}\eta_{Y*}\epsilon_A = \eta_{X*}\pi_{XY*}\epsilon_A = 0$, it follows that $\eta_{Y*}\epsilon_A \in V(T_{\eta_Y(y)}Y)$ and thus $\eta_{L_V Y} \in \text{Aut}(L_V Y)$. Therefore, $\eta_{L_V Y}$ is the canonical lift of η_Y . The $(n+k)$ -symplectic potential $i^*\theta$ on $L_V Y$ is invariant under the canonical lift of each automorphism of Y . Conversely, every element of $\text{Aut}(L_V Y)$ that leaves $i^*\theta$ invariant is a lift of an automorphism of Y . To justify this, we merely take a result in [9] for lifts of $\text{Diff } M$ to LM and modify it for lifts of $\text{Aut } Y$ to $L_V Y$.

Remark We cannot naturally define a lift of $\text{Diff } Y$ to $\text{Aut}(L_V Y)$. Indeed, let $X = \mathbb{R}$ and $Y = X \times \mathbb{R}$. Define $f \in \text{Diff } Y$ by $f(x, y) = (y, x)$. Then $f_*(\frac{\partial}{\partial y}) = \frac{\partial}{\partial x}$, which implies that $f_*(V(TY)) \not\subseteq V(TY)$.

Denote the action of $\text{Aut } Y$ on Y by $\Psi : \text{Aut } Y \times Y \rightarrow Y$ and the lifted $(n+k)$ -symplectic action on $L_V Y$ by $\tilde{\Psi}$. A projectable vector field generates a one-parameter group ψ_t of local automorphisms of Y , which lifts to a one-parameter group $\tilde{\psi}_t$ of elements of $\text{Aut}(U)$ on some open set $U \subset L_V Y$. The vector field $\xi_U := (d/dt)\tilde{\psi}_t$ on U is the infinitesimal generator of $\tilde{\psi}_t$ and thus is the natural lift of $\xi_{\pi_{XY}(U)}$.

Proposition 5.1 *Let $\xi \in \mathcal{X}_{\text{Proj}} Y$ be an infinitesimal generator of the action $\tilde{\Psi}$. Then*

$$\hat{J}(\xi) := \xi_{L_V Y} \lrcorner i^*\theta$$

is in $T_V^1(L_V Y)$, and its corresponding momentum mapping J is Ad^ equivariant. Furthermore, every element of $T_V^1(L_V Y)$ can be identified with $\hat{J}(\xi)$ for some $\xi \in \mathcal{X}_{\text{Proj}} Y$.*

Proof Because $i^*\theta$ is invariant under $\tilde{\Psi}$, it follows that $\mathbf{L}_{\xi_{L_V Y}} i^*\theta = 0$. From (2.7), $d(\xi_{L_V Y} \lrcorner i^*\theta) = -\xi_{L_V Y} \lrcorner i^*d\theta$, so $\xi_{L_V Y} \lrcorner i^*\theta \in HF^1(L_V Y)$. If $\tilde{\psi}_t$ is the local flow of $\xi_{L_V Y}$ in a neighborhood of $w \in L_V Y$, then for $g \in G_{\mathcal{A}}$, $R_g \circ \tilde{\psi}_t = \tilde{\psi}_t \circ R_g$. Thus, $R_{g*} \xi_{L_V Y}(w) = \xi_{L_V Y}(R_g(w))$. Because $i^*\theta$ is $G_{\mathcal{A}}$ -tensorial, it now follows that $R_g^*(\xi_{L_V Y} \lrcorner i^*\theta)(w) = g^{-1} \cdot (\xi_{L_V Y} \lrcorner i^*\theta)(w)$. So $\xi_{L_V Y} \lrcorner i^*\theta \in T_V^1(L_V Y)$. The proof that J is Ad^* equivariant is analogous to [1, Thm. 4.2.10]. By Proposition 4.1 any element of $T_V^1(L_V Y)$ can be identified with some ξ in $\mathcal{X}_{\text{Proj}} Y$ and thus with $\hat{J}(\xi)$. \square

For $\xi, \zeta \in \mathcal{X}_{\text{Proj}} Y$, define an \mathbb{R}^{n+k} -valued Poisson bracket by

$$\{\hat{J}(\xi), \hat{J}(\zeta)\}^\mu = -di^*\theta^\mu(\xi_{L_V Y}, \zeta_{L_V Y}), \quad \mu = 1, 2, \dots, n+k. \quad (5.1)$$

By equation (5.1), identities (2.7), and the fact that $\pi_{*Y} L_V Y[\xi_{L_V Y}, \zeta_{L_V Y}] = [\xi_Y, \zeta_Y]$,

$$\{\hat{J}(\xi), \hat{J}(\zeta)\} = \hat{J}([\xi, \zeta]). \quad (5.2)$$

Observe that (5.2) has no additional term, in contrast to (2.8).

6 Recovering multisymplectic momentum mappings

Define a linear left action of $G_{\mathcal{A}}$ on the vector space $\mathbb{R}^{n \times k} \times \mathbb{R}$ as follows.

$$(N, K, A) \cdot (B, \lambda) := \det(N^{-1}) (NBK^{-1}, \lambda - \text{tr}(BK^{-1}A)) \quad (6.1)$$

The associated vector bundle $L_V Y \times_{G_{\mathcal{A}}} (\mathbb{R}^{n \times k} \times \mathbb{R})$ is constructed using the actions in (4.1) and (6.1). For a point $(y, \{e_i, \epsilon_A\}) \in L_V Y$, define

$$\omega(e) := e^1 \wedge e^2 \wedge \dots \wedge e^n \quad \text{and} \quad \omega(e)_i := e_i \lrcorner \omega(e), \quad (6.2)$$

where $\{e^i\}$ is the dual basis of $\{e_i\}$. From (4.2), if $(y, \{e'_i, \epsilon'_A\}) = (y, \{e_i, \epsilon_A\}) \cdot (N, K, A)$ then $\omega(e') = \det(N^{-1})\omega(e)$ where $e = \{e_i\}$ and $e' = \{e'_i\}$.

The map

$$\begin{aligned} \hat{\rho} : L_V Y \times_{G_{\mathcal{A}}} (\mathbb{R}^{n \times k} \times \mathbb{R}) &\rightarrow \wedge^n Y \\ [(y, \{e_i, \epsilon_A\}), (B, \lambda)] &\mapsto (y, B_A^i \epsilon^A \wedge \omega(e)_i + \lambda \omega(e)) \end{aligned} \quad (6.3)$$

is a vector bundle monomorphism over Y , and the range of $\hat{\rho}$ is Z . Thus, the multisymplectic phase space Z is a vector bundle over Y associated to $L_V Y$. The proof of this result appears in [10]. The relationships between local coordinates $\{x^i, y^A, p_B^j, p\}$ on Z and $\{x^i, y^A, \pi_C^j, \pi_l^B, \pi_l^D\}$ on $L_V Y$ are

$$p_B^j = \det(\pi_b^a) B_A^i \pi_B^A (\pi^{-1})_i^j \quad \text{and} \quad p = \det(\pi_b^a) (B_A^i \pi_k^A (\pi^{-1})_i^k + \lambda).$$

Let $B \in \mathbb{R}^{n \times k}$ and $\lambda \in \mathbb{R}$. Define the map

$$\phi_{(B, \lambda)} : L_V Y \rightarrow Z : w \mapsto \hat{\rho}[w, (B, \lambda)]. \quad (6.4)$$

The map $\phi_{(B,\lambda)}$ preserves fibers over Y , and its range is a subbundle of Z with standard fiber the G_A -orbit of (B, λ) . The G_A -orbits for all nonzero $B \in \mathbb{R}^{n \times k}$ are classified by the rank of B . Using Proposition 4.1 we may establish that for $\xi \in \mathcal{X}_{\text{Proj}} Y$,

$$\phi_{(B,\lambda)*} \xi_{L_V Y} = \xi_Z. \quad (6.5)$$

This is analogous to [16, Thm. 5.2] and is easily verified by local coordinate calculations.

Lemma 6.1 *For $n \geq 2$, the $\wedge^n \mathbb{R}^{n+k}$ -valued n -form $i^* \wedge^n \theta$ on $L_V Y$ can be related to the canonical n -form Θ on Z by*

$$\langle \wedge^n i^* \theta, \mathbf{V}(B, \lambda) \rangle = \phi_{(B,\lambda)}^* \Theta \quad (6.6)$$

where the map $\mathbf{V} : \mathbb{R}^{n \times k} \times \mathbb{R} \rightarrow \wedge^n \mathbb{R}^{n+k*}$ has components

$$\begin{aligned} V_{i_1 \dots i_n}(B, \lambda) &= \frac{1}{n!} \lambda \epsilon_{i_1 \dots i_n}, \quad V_{A i_1 \dots i_{n-1}}(B, \lambda) = \frac{1}{n!} B_A^j \epsilon_{j i_1 \dots i_{n-1}} \\ \text{and} \quad V_{A_1 \dots A_l i_1 \dots i_{n-l}}(B, \lambda) &= 0 \quad \forall l \geq 2. \end{aligned}$$

The proof of this lemma is in [10]. Observe that (6.6) holds for $n = 1$ if $V_i(B, \lambda) = \lambda$ and $V_A(B, \lambda) = B_A$.

Lemma 6.2 *Let $\bar{\Psi}$ be the lift to Z of the action Ψ . Let $B \in \mathbb{R}^{n \times k}$ and $\lambda \in \mathbb{R}$. Then, using (6.3),*

$$\bar{\Psi}(\eta_Y, [(y, \{e_i, \epsilon_A\}), (B, \lambda)]) \simeq \left[\tilde{\Psi}(\eta_Y, (y, \{e_i, \epsilon_A\})), (B, \lambda) \right].$$

Consequently, $\phi_{(B,\lambda)} \circ \tilde{\Psi}_f(w) = \bar{\Psi}_f \circ \phi_{(B,\lambda)}(w)$.

Proof Let $(y, \{e_i, \epsilon_A\}) \in L_V Y$. From (6.2),

$$\omega(e) := \frac{1}{n!} \epsilon_{i_1 \dots i_n} e^{i_1} \wedge \dots \wedge e^{i_n} \quad \text{and} \quad \omega(e)_j := \frac{1}{(n-1)!} \epsilon_{j i_1 \dots i_{n-1}} e^{i_1} \wedge \dots \wedge e^{i_{n-1}},$$

where $\epsilon_{i_1 \dots i_n}$ is the sign of the permutation $(1, \dots, n) \mapsto (i_1, \dots, i_n)$. Using (6.1) and (6.3),

$$\begin{aligned} \left[\tilde{\Psi}(\eta_Y, (y, \{e_i, \epsilon_A\})), (B, \lambda) \right] &\simeq (\eta_Y(y), B_A^i (\eta_{Y*} \epsilon)^{*A} \wedge \omega(\eta_{Y*} e)_i + \lambda \omega(\eta_{Y*} e)) \\ &= (\eta_Y(y), \eta_Y^{-1*} (B_A^i \epsilon^A \wedge \omega(e)_i + \lambda \omega(e))) \\ &\simeq \bar{\Psi}(\eta_Y, [(y, \{e_i, \epsilon_A\}), (B, \lambda)]). \end{aligned}$$

By choosing representatives of the equivalence classes, we see that $\phi_{(B,\lambda)} \circ \tilde{\Psi}_f(w) = \hat{\Psi}_f \circ \phi_{(B,\lambda)}(w)$. \square

Let $\{R^\mu\} \mu = 1, \dots, n+k$, be the basis of \mathbb{R}^{n+k*} dual to $\{R_\mu\}$. Define the $\wedge^{r+s} \mathbb{R}^{n+k}$ -valued $(p+q)$ -form $R_{\mu_1 \dots \mu_m} := R_{\mu_1} \wedge \dots \wedge R_{\mu_m} \in \wedge^m \mathbb{R}^{n+k}$ and $R^{\mu_1 \dots \mu_m} := R^{\mu_1} \wedge \dots \wedge R^{\mu_m} \in \wedge^m \mathbb{R}^{n+k*}$. Let α be a $\wedge^r \mathbb{R}^{n+k}$ -valued p -form and let β be a $\wedge^s \mathbb{R}^{n+k}$ -valued q -form on a manifold. Then $\alpha = \alpha^{\mu_1 \dots \mu_r} \otimes R_{\mu_1 \dots \mu_r}$ and $\beta = \beta^{\mu_1 \dots \mu_s} \otimes R_{\mu_1 \dots \mu_s}$. Define

$$\alpha \wedge \beta := (\alpha^{\mu_1 \dots \mu_r} \wedge \beta^{\nu_1 \dots \nu_s}) \otimes R_{\mu_1 \dots \mu_r \nu_1 \dots \nu_s}.$$

Theorem 6.3 *Let $w \in L_V Y$ and $(B, \lambda) \in \mathbb{R}^{n \times k} \times \mathbb{R}$.*

$$\phi_{(B, \lambda)}^*(\hat{J}_Z(\xi)) = \left\langle \hat{J}_{L_V Y}(\xi) \wedge (\wedge^{n-1} i^* \theta), n \mathbf{V}(B, \lambda) \right\rangle$$

where $\mathbf{V}(B, \lambda) \in \wedge^n \mathbb{R}^{(n+k)*}$ is defined in Lemma 6.1.

Proof Let ψ_t be a local one-parameter group of X , and let $\tilde{\psi}_t$ on $L_V Y$ and $\bar{\psi}_t$ on Z be the respective lifts. It follows from Lemma 6.2 and equation (6.5) that $\phi_{(B, \lambda)} \circ \tilde{\psi}_t = \bar{\psi}_t \circ \phi_{(B, \lambda)}$, and thus $\phi_{(B, \lambda)*} \xi_{L_V Y} = \xi_Z \circ \phi_{(B, \lambda)}$. Using Lemma 6.1 and equation (2.3),

$$\begin{aligned} \left\langle \hat{J}_{L_V Y}(\xi) \wedge (\wedge^{n-1} i^* \theta), n \mathbf{V}(B, \lambda) \right\rangle &= \langle (\xi_{L_V Y} \lrcorner \wedge^n i^* \theta), \mathbf{V}(B, \lambda) \rangle \\ &= \xi_{L_V Y} \lrcorner \phi_{(B, \lambda)}^* \Theta \\ &= \phi_{(B, \lambda)}^* (\xi_Z \lrcorner \Theta) \\ &= \phi_{(B, \lambda)}^* \hat{J}_Z(\xi). \quad \square \end{aligned}$$

The last theorem motivates the representation of $T_V^1(L_V Y)$ in the space of $\wedge^{m+1} \mathbb{R}^{n+k}$ -valued m -forms by

$$\hat{J}(\xi) \mapsto \hat{J}(\xi) \wedge (\wedge^m i^* \theta) \quad (6.7)$$

for $0 \leq m \leq n + k$. By induction,

$$\xi_{L_V Y} \lrcorner \wedge^m i^* \theta = m \hat{J}(\xi) \wedge (\wedge^{m-1} i^* \theta). \quad (6.8)$$

For each $0 \leq m \leq n + k - 2$ the form $d(\wedge^{m+1} i^* \theta)$ is closed and nondegenerate. Using (6.8) and

$$\text{convd}(\wedge^m \theta) = m d\theta \wedge (\wedge^{m-1} \theta),$$

it follows that

$$d(\hat{J}(\xi) \wedge (\wedge^m i^* \theta)) = -\xi_{L_V Y} \lrcorner (i^* d\theta \wedge (\wedge^m i^* \theta)).$$

For $0 \leq m \leq n + k$ define a bracket on the image of representation (6.7) by

$$\{\hat{J}(\xi) \wedge (\wedge^m i^* \theta), \hat{J}(\zeta) \wedge (\wedge^m i^* \theta)\} := \xi_{L_V Y} \lrcorner \zeta_{L_V Y} \lrcorner (i^* d\theta \wedge (\wedge^m i^* \theta)).$$

If $m = 0$ then this becomes the bracket on $T_V^1(L_V Y)$ given by (5.1). This sign convention for the bracket is consistent with [6] and is the negative of that of [15]. Computation on the forms yields, for $0 \leq m \leq n + k$,

$$\begin{aligned} \{\hat{J}(\xi) \wedge (\wedge^m i^* \theta), \hat{J}(\zeta) \wedge (\wedge^m i^* \theta)\} \\ = \{\hat{J}(\xi), \hat{J}(\zeta)\} \wedge (\wedge^m i^* \theta) + m d(\hat{J}(\xi) \wedge \hat{J}(\zeta) \wedge (\wedge^{m-1} i^* \theta)). \end{aligned} \quad (6.9)$$

Equation (6.9) is of particular interest when $m = n - 1$. In this case we have reproduced on $L_V Y$ the result in (2.8) that $T^1(Z)$ is not closed under its bracket operation. But recall that $T_V^1(L_V Y)$ already possesses a well-defined Lie algebra under (5.1), and by (6.5) an infinitesimal generator of the lifted action of $\text{Aut } Y$ on $L_V Y$ pushes forward to a generator of the lifted action on Z . Thus Proposition 4.1 and equations (5.2) and (6.5) combine to demonstrate that the $(n+k)$ -symplectic momentum observables of $L_V Y$ not only generalize the momentum observables on Z but also possess an algebraic structure that is absent on Z .

7 Conserved quantities

Let $ST^2(L_V Y)$ denote the vector space of tensorial $\mathbb{R}^{n+k} \otimes_s \mathbb{R}^{n+k}$ -valued functions on $L_V Y$. Let $HF^2(L_V Y)$ denote the vector space of $\mathbb{R}^{n+k} \otimes_s \mathbb{R}^{n+k}$ -valued symmetric Hamiltonian observables, namely, functions $\hat{g} = \hat{g}^{\mu\nu} S_{\mu\nu}$ on $L_V Y$ that satisfy the equation

$$d\hat{g}^{\mu\nu} = -2X^{(\mu} \lrcorner i^* d\theta^{\nu)}, \quad (7.1)$$

where $X^\mu R_\mu$ is an \mathbb{R}^{n+k} -valued Hamiltonian vector field and $S_{\mu\nu} := R_\mu \otimes_s R_\nu$. Finally, let $ST_V^2(L_V Y) := ST^2(L_V Y) \cap HF^2(L_V Y)$.

A function $\hat{g} \in ST^2(L_V Y)$ appears in local coordinates on $L_V Y$ as

$$\begin{aligned} \hat{g}(u) = & g^{ij} \pi_i^k \pi_j^l S_{kl} + g^{ij} \pi_i^A \pi_j^l S_{lA} + g^{iA} \pi_i^l \pi_A^B S_{lB} \\ & + g^{ij} \pi_i^A \pi_j^B S_{AB} + g^{iC} \pi_i^A \pi_C^B S_{AB} + g^{CD} \pi_C^A \pi_D^B S_{AB}, \end{aligned} \quad (7.2)$$

where the component functions g^{ij} , g^{iA} , and g^{AB} are functions of x^k and y^A . If $\hat{g} \in ST_V^2(L_V Y)$ then the only additional restriction is that $g^{ij} = g^{ij}(x^k)$. So $ST_V^2(L_V Y)$ is in bijective correspondence with the space of projectable symmetric tensors of degree 2 on Y . This extends Proposition 4.1 to degree 2.

As in the theory on LM [15], symmetrization in equation (7.1) means that the \mathbb{R}^{n+k} -valued Hamiltonian vector fields $X_{\hat{g}}^\mu R_\mu$ are not uniquely determined by $\hat{g} \in ST_V^2(L_V Y)$. Rather, they are determined locally up to the addition of vector fields $Y^\mu R_\mu$ which satisfy

$$Y^{(\mu} \lrcorner i^* d\theta^{\nu)} = 0. \quad (7.3)$$

Thus each Y^μ must be a vertical vector field. For a given $\hat{g} \in ST_V^2(L_V Y)$, two Hamiltonian vector fields are in the same equivalence class $\llbracket X_{\hat{f}} \rrbracket^\mu R_\mu$ if their difference Y^μ satisfies (7.3). Obtaining vector fields only up to equivalence does not affect the basic $(n+k)$ -symplectic algebraic structures on $L_V Y$. For example, if $\hat{f} \in T_V^1(L_V Y)$ and $\hat{g} \in ST_V^2(L_V Y)$, then define a Poisson bracket by

$$\{\hat{f}, \hat{g}\} := -X_{\hat{f}}(\hat{g}^{\mu\nu}) S_{\mu\nu}$$

or use any representative of the equivalence class $\llbracket X_{\hat{g}} \rrbracket^\mu$ to define a Poisson bracket by

$$\{\hat{g}, \hat{f}\} = -2X_{\hat{g}}^{(\mu}(\hat{f}^{\nu)}) S_{\mu\nu}.$$

As a consequence, $\{\hat{g}, \hat{f}\} = -\{\hat{f}, \hat{g}\}$.

The proof of the following lemma and theorem are given by Norris [16] for LY and are easily modified for $L_V Y$. For the theorem, Norris examines actions of subgroups of $\text{Diff } Y$, but we may modify his proof to accommodate actions of subgroups of $\text{Aut } Y$.

Lemma 7.1 *Let $\hat{f} = \hat{f}^\mu R_\mu \in T_V^1(L_V Y)$ and let $X_{\hat{f}}$ be its corresponding Hamiltonian vector field. Let $\hat{g} \in ST_V^2(L_V Y)$ and let $X_{\hat{g}}$ be a representative of its equivalence class $\llbracket X_{\hat{g}} \rrbracket$ of corresponding \mathbb{R}^{n+k} -valued Hamiltonian vector fields. If $\{\hat{g}, \hat{f}\} = 0$, then for each $\mu = 1, 2, \dots, n+k$, \hat{f}^μ is constant on the orbits of each $X_{\hat{g}}^\mu \in \llbracket X_{\hat{g}} \rrbracket^\mu$.*

Theorem 7.2 *Let Φ be an $(n+k)$ -symplectic action of a subgroup H of $\text{Aut } Y$ upon $L_V Y$, and let J be the corresponding momentum mapping. Suppose that $\hat{g} \in ST_V^2(L_V Y)$ is invariant under the action. Then J provides $n+k$ integrals of \hat{g} in the sense that*

$$J^\mu(F_t^\mu(u)) = J^\mu(u),$$

where F_t^μ is the flow of any representative of $\llbracket X_{\hat{g}} \rrbracket^\mu$, for $\mu = 1, 2, \dots, n+k$.

Momentum mappings on $L_V Y$ are not dissimilar to those on LY , but the following examples on $L_V Y$ illustrate the advantage of the inherent vertical adaptation.

8 Applications of momentum mappings

8.1 Angular momentum in a field theory

We use momentum mappings on $L_V Y$ to express angular momentum in a field theory, thus improving upon examples on T^*M [1] and on LM [16]. Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^{n+k}$, and let $\{x^i, y^A\}$ be adapted coordinates on Y . Let η be a flat metric on \mathbb{R}^n and ι be a flat metric on \mathbb{R}^k . Assume an Ehresmann connection $\gamma : TY \rightarrow V(TY)$. The **Kaluza-Klein metric** is the fiberwise bilinear map G on TY ,

$$G(v, w) = \pi_{XY}^* \eta(v, w) + \iota(\gamma(v), \gamma(w)).$$

If we use the trivial Ehresmann connection, then in adapted coordinates,

$$G(v, w) = \eta_{ij} v^i w^j + \iota_{AB} v^A w^B.$$

Let $\mathcal{O}_\eta(n)$ be the orthogonal group with respect to the metric η and let $\mathcal{O}_\iota(k)$ be the orthogonal group with respect to ι . Define a left action $\Phi : (\mathcal{O}_\eta(n) \times \mathcal{O}_\iota(k)) \times Y \rightarrow Y$ by left matrix multiplication on column vectors in \mathbb{R}^{n+k} , and $\Phi_g(y) := \Phi(g, y)$. Define a left action

$$\underline{\Phi} : (\mathcal{O}_\eta(n) \times \mathcal{O}_\iota(k)) \times X \rightarrow X$$

by $\underline{\Phi}(g, x) = \pi \circ \Phi(g, \hat{x}) =: \underline{\Phi}_g(x)$ where $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k} : x \mapsto \hat{x}$ is inclusion. Because Φ_g and $\underline{\Phi}_g$ are bijections and $\underline{\Phi}_g \circ \pi = \pi \circ \Phi_g$, it follows that $g \mapsto \underline{\Phi}_g$ is a representation of $\mathcal{O}_\eta(n) \times \mathcal{O}_\iota(k)$ in $\text{Aut}(Y)$.

Remark The larger group $O_G(n+k)$ does not possess a nontrivial left action on $X = \mathbb{R}^n$. Under left multiplication of $GL(n+k)$ on \mathbb{R}^{n+k} , $g \in GL(n+k)$ leaves \mathbb{R}^n invariant if and only if $g \in G_A$. Thus, we use $\mathcal{O}_\eta(n) \times \mathcal{O}_\iota(k) = G_A \cap O_G(n+k)$.

To compute the infinitesimal generators, let $\xi = \xi_\nu^\mu E_\mu^\nu \in \mathfrak{o}_\eta(n) \times \mathfrak{o}_\iota(k) \subset \mathfrak{gl}(n+k)$, where $\{E_\mu^\nu\}$ is the standard basis for the Lie algebra $\mathfrak{gl}(n+k)$ and $\{E_j^i, E_B^A\}$ is a basis for $\mathfrak{o}_\eta(n) \times \mathfrak{o}_\iota(k)$. Then

$$\xi_Y(y) = \xi_j^i x^j \frac{\partial}{\partial x^i} + \xi_B^A y^B \frac{\partial}{\partial y^A}.$$

Note that $\pi_{XY*}(\xi_Y(y)) = \xi_X(\pi_{XY*}(y))$. Now, ξ_Y is a **Killing vector field** of G . Indeed, it is easy to verify that

$$G_{\mu\nu}(\xi_Y)^\nu{}_{,\kappa} + G_{\kappa\nu}(\xi_Y)^\nu{}_{,\mu} = 0. \quad (8.1)$$

Let $u \in L_V Y$ and $\xi \in \mathfrak{o}_\eta(n) \times \mathfrak{o}_\iota(k)$. The momentum mapping on $L_V Y$ associated with the action defined above is

$$\hat{J}_{L_V Y}(\xi)(u) = \xi_k^j x^k \pi_j^i \hat{r}_i + (\xi_k^j x^k \pi_j^A + \xi_C^B y^C \pi_B^A) \hat{s}_A,$$

and if $\{C_\nu^\mu\}$ is the basis of $\mathfrak{gl}(n+k)^*$ dual to $\{E_\mu^\nu\}$ then

$$\begin{aligned} J(u) &= C_k^j x^k \pi_j^i \hat{r}_i + (C_k^j x^k \pi_j^A + C_D^B y^D \pi_B^A) \hat{s}_A \\ &= \eta_{kl} C^{lj} x^k \pi_j^i \hat{r}_i + (\eta_{kl} C^{jl} x^k \pi_j^A + \iota_{DE} C^{DB} y^E \pi_B^A) \hat{s}_A \\ &= \left(x_{[i} \pi_{j]}^k C^{ji} \hat{r}_k + \left(x_{[i} \pi_{j]}^A C^{ji} + y_{[D} \pi_{B]}^A C^{BD} \right) \hat{s}_A \right). \end{aligned} \quad (8.2)$$

The function $\hat{G} \in ST_V^2(L_V Y)$ corresponds to the metric G and represents the energy tensor. Via equation (7.1), \hat{G} produces an \mathbb{R}^{n+k} -valued Hamiltonian vector field, whose components are

$$X_G^i = \eta^{jk} \pi_j^i \frac{\partial}{\partial x^k} \quad \text{and} \quad X_G^A = \eta^{jk} \pi_j^A \frac{\partial}{\partial x^k} + \iota^{BC} \pi_B^A \frac{\partial}{\partial y^C}. \quad (8.3)$$

Equation (8.1) implies that the X_G^μ are tangent to the $O_\eta(n) \times O_\iota(k)$ subbundle of $L_V Y$.

By Theorem 7.2, each J^μ is constant along X_G^μ for the same μ . We may interpret J^i to be the i^{th} component of the (extrinsic) angular momentum in parameter space (e.g., Minkowski spacetime) and J^A to be the A^{th} component of the total angular momentum (both extrinsic and field) in the field configuration space. As a result, we may integrate the first equation in (8.3) to obtain that

$$F_t^i(x^k, y^A, \pi_l^m, \pi_n^B, \pi_D^C) = (x^k + t\eta^{kj} \pi_j^i, y^A, \pi_l^m, \pi_n^B, \pi_D^C)$$

is the flow of X_G^i on X . Likewise, The flow for X_G^A is

$$F_t^A(x^k, y^A, \pi_l^m, \pi_n^B, \pi_D^C) = (x^k + t\eta^{kj} \pi_j^A, y^C + t\iota^{CB} \pi_B^A, \pi_l^m, \pi_n^D, \pi_F^E),$$

In fact,

$$J^i \circ F_t^i = \dot{x}^{[l} x^{k]} C_{lk} = J^i \quad (\text{no sum on } i) \quad (8.4)$$

and

$$J^A \circ F_t^A = \dot{x}^{[l} x^{k]} C_{lk} + \dot{y}^{[D} y^{B]} C_{DB} = J^A \quad (\text{no sum on } A) \quad (8.5)$$

which is consistent with Theorem 7.2. From (8.4) we may obtain n independent conserved quantities, corresponding to conservation of extrinsic angular momentum in n independent directions in parameter spacetime. Similarly, from (8.5) we may obtain conservation of total angular momentum in the $n+k$ dimensions of the field theory.

If $n = 1$ then we may model time-evolution particle mechanics on (\mathbb{R}^k, ι) . The group action is up to linear time rescaling, merely that of $O_\iota(k)$ on \mathbb{R}^k . Equation (8.4) is trivial and (8.5) reduces to $J^A \circ F_t^A = \dot{y}^{[A} y^{B]} C_{AB} = J^A$. Thus, time rescaling has no effect on conservation of angular momentum. Likewise, if $k = 1$ then we may model scalar fields over parameter space (\mathbb{R}^n, η) . The group action is, up to linear rescaling in the field, merely that of $O_\eta(n)$ on \mathbb{R}^n . Then (8.4) is left intact and (8.5) reduces to (8.4). Thus, the scalar field angular momentum comes purely from parameter space.

8.2 Affine symmetry in time-evolution mechanics

We'll examine the previous example more closely in the case where $n = 1$, extending an example found in [10]. Local coordinates will be $\{x^0 = t, y^A\}$. and, again we assume a trivial connection of $Y = \mathbb{R}^{1+k}$ over $X = \mathbb{R}$ and construct the Kaluza-Klein metric G on Y ,

$$G(v, w) = v^0 w^0 + \iota_{AB} v^A w^B.$$

Construct a subgroup of $GL(k+1)$ defined by

$$\mathcal{G}(k) := \left\{ \begin{pmatrix} 1 & 0 \\ v & K \end{pmatrix} \middle| K \in O_\iota(k), v \in \mathbb{R}^k \right\}.$$

The group $\mathcal{G}(k)$ is the semidirect product of \mathbb{R}^k with $O_\iota(k)$. The left representation $\Phi : \mathcal{G}(k) \times Y \rightarrow Y$ defined by left matrix multiplication covers the trivial representation of $\mathcal{G}(k)$ on $X = \mathbb{R}$, so that for all $g \in \mathcal{G}(k)$, $\Phi_g \in \text{Aut}(Y)$. Lift Φ_g to an action of $\mathcal{G}(k)$ on $L_V Y$. The corresponding Lie algebra $\mathfrak{g}(k)$ is the semidirect product of \mathbb{R}^k with $\mathfrak{o}_\iota(k)$, and may be represented as a subalgebra of $\mathfrak{gl}(k+1)$ by

$$\mathfrak{g}(k) = \left\{ \begin{pmatrix} 0 & 0 \\ v & k \end{pmatrix} \middle| k \in \mathfrak{o}_\iota(k), v \in \mathbb{R}^k \right\}$$

If we express the basis of $\mathfrak{g}(k)$ by $\{E_B^A, s_C\}$ where $\{E_B^A\}$ is the standard basis for $\mathfrak{o}_\iota(k)$ and $\{s_C\}$ is the standard basis for \mathbb{R}^k . For $\xi \in \mathfrak{g}(k)$, we may write $\xi = k_B^A E_A^B + v^A s_A$. In local coordinates the corresponding infinitesimal generator on Y is $\xi_Y(y) = (k_B^A y^B + x^0 v^A) \frac{\partial}{\partial y^A}$. The corresponding momentum mapping on $L_V Y$ is

$$\hat{J}(\xi)(u) = (k_C^A y^C \pi_A^B + x^0 v^A \pi_A^B) \hat{s}_B$$

or if $\{C_B^A, s^D\}$ is the basis of $\mathfrak{g}(k)^*$ dual to $\{E_B^A, s_C\}$,

$$J(u) = (y^D \pi_A^B C_D^A + x^0 \pi_A^B s^A) \hat{s}_B = (y_{[D} \pi_{A]}^B C^{AD} + x^0 \pi_A^B s^A) \hat{s}_B.$$

The function \hat{G} corresponding to the metric G is in $ST_V^2(L_V Y)$ and, via equation (7.1), produces an equivalence class of \mathbb{R}^{1+k} -valued Hamiltonian vector fields $[[X_{\hat{G}}]]$. We choose a representative of this equivalence class, expressed in local coordinates as

$$X_{\hat{G}}^0 = \pi_0^0 \frac{\partial}{\partial x^0} \quad \text{and} \quad X_{\hat{G}}^A = \pi_0^A \frac{\partial}{\partial x^0} + \iota^{BC} \pi_B^A \frac{\partial}{\partial y^C}.$$

Along an integral curve of $X_{\hat{G}}^0$, we get the equations $\dot{x}^0 = \pi_0^0$ and $\dot{y}^A = \dot{\pi}_C^B = \dot{\pi}_0^D = \dot{\pi}_0^0 = 0$. Integrating with suitable initial conditions we obtain $F_\lambda^0 = (x^0 + \pi_0^0 \lambda, y^A, \pi_C^B, \pi_0^D, \pi_0^0)$. So,

$$J^0 \circ F_\lambda^0 = J^0 = 0.$$

Along an integral curve of $X_{\hat{G}}^A$, we obtain the equations

$$\dot{x}^0 = \pi_0^A, \quad \dot{y}^C = \iota^{BC} \pi_B^A, \quad \text{and} \quad \dot{\pi}_C^B = \dot{\pi}_0^D = \dot{\pi}_0^0 = 0.$$

Integrating with initial conditions we get $F_\lambda^A = (x^0 + \pi_0^A \lambda, y^C + \lambda \iota^{BC} \pi_B^A, \pi_D^B, \pi_0^E, \pi_0^0)$. So,

$$\begin{aligned} J^B \circ F_\lambda^B &= (x^0 + \pi_0^B \lambda) \pi_A^B s^A + (y^C + \lambda \iota^{CE} \pi_C^B) \pi_D^B C_E^D \quad \text{no sum on } B \\ &= x^0 \dot{y}^C \iota_{AC} s^A + y^E \dot{y}^D \iota_{AD} C_E^A + \lambda \dot{x}^0 \iota_{AC} \dot{y}^C s^A \\ &= y_{[E} \dot{y}_{D]} C^{DE} + x^0 \dot{y}^C \iota_{AC} s^A + \lambda \dot{x}^0 \iota_{AC} \dot{y}^C s^A \\ &= J^B + \lambda \dot{x}^0 \iota_{AC} \dot{y}^C s^A. \end{aligned} \tag{8.6}$$

Apparently (8.6) violates conservation of angular momentum along the flow, because F_λ^B is *not* an integral of the motion. In fact, the additional term describes the contribution to the angular momentum from shift of the point in Y under the translational part of $\mathcal{G}(k)$. Indeed, if the action is restricted to $O_\iota(k) \subset \mathcal{G}(k)$, then the extra term disappears as in equation (8.5), and again angular momentum is conserved along flows in the spatial directions. Thus the $(n+k)$ -symplectic geometry adjusts the angular momentum to accommodate a change in reference frame.

If $k = 3$ and ι is the Euclidean metric on \mathbb{R}^3 , then $\mathcal{G}(3) = E(3)$, the Euclidean group. Equation (8.6) may be interpreted to be the ***parallel axis theorem*** of classical rigid body mechanics. If $k = 4$ and ι is the Lorentz metric, then $\mathcal{G}(4) = E(4)$, the Poincaré group, and (8.6) indicates how to transform 4-angular momentum when boosting from one relativistic inertial reference frame to another.

8.3 Time reparametrization symmetry in time-evolution mechanics

Let $X = \mathbb{R}$ and let $Y = \mathbb{R} \times Q$, where (Q, g) is a k -dimensional Riemannian manifold. Local coordinates on Y are $\{x^0 = t, y^A\}$. Assume a generic connection $\gamma : TY \rightarrow V(TY)$ expressed locally by $\gamma(v) = (v^A - v^0 \gamma^A) \frac{\partial}{\partial y^A}$. The resulting Kaluza-Klein metric G on Y has local coordinate expression,

$$G(v, w) = (1 + g_{AB} \gamma^A \gamma^B) v^0 w^0 + g_{AB} \gamma^A (v^B w^0 + v^0 w^B) + g_{AB} v^A w^B.$$

Let $\mathcal{G} = \text{Diff } \mathbb{R}$ act on Y by time reparametrizations. The corresponding Lie algebra \mathfrak{g} may be identified with $C^\infty(\mathbb{R})$, and we may identify \mathfrak{g}^* , the space of densities on \mathbb{R} , with \mathfrak{g} . This action may be represented in $\text{Aut } Y$, as it leaves the fiber of Y constant, and thus the action lifts to $L_V Y$. In local coordinates, $f \in \text{Diff } \mathbb{R}$ has infinitesimal generator $f_Y = f(x^0) \frac{\partial}{\partial x^0}$, which lifts to a projectable vector field on $L_V Y$, $f_{L_V Y} = f(x^0) \frac{\partial}{\partial x^0}$. Using Proposition 5.1 the corresponding momentum observable is

$$\hat{J}_{L_V Y}(f) = f(x^0) \frac{\partial}{\partial x^0} \lrcorner ((\pi_B^A dy^B + \pi_0^A dx^0) \hat{s}_A + \pi_0^0 dx^0 \hat{r}_0) = f(x^0) (\pi_0^A \hat{s}_A + \pi_0^0 \hat{r}_0)$$

or

$$J_{L_V Y} : f \mapsto f \cdot (\pi_0^A \hat{s}_A + \pi_0^0 \hat{r}_0).$$

On $L_V Y$ we may note the time-dependence of the momentum observables. In fact the second term describes momentum along the time parameter and the first term provides an adjustment in momentum in the fiber of Y to account for the time reparametrization.

By (7.1), The tensorial function $\hat{G} \in ST_V^2(L_V Y)$ corresponding to the Kaluza-Klein metric G produces equivalence classes of \mathbb{R}^{1+k} -valued Hamiltonian vector fields $\llbracket X_{\hat{G}} \rrbracket^\mu$. If in addition, a Hamiltonian vector field is required to satisfy a “no-torsion” condition,

$$X_g^\mu \lrcorner X_g^\nu \lrcorner i^* d\theta = 0,$$

then the Hamiltonian vector field is unique and may be expressed locally as

$$X_g^\mu = G^{\nu\lambda} \pi_\nu^\mu \frac{\partial}{\partial Y^\lambda} + \Gamma_{\sigma\rho}^\nu G^{\kappa\rho} \pi_\kappa^\mu \pi_\nu^\lambda \frac{\partial}{\partial \pi_\sigma^\lambda} \quad (8.7)$$

where $\{Y^\mu\} = \{x^0, y^A\}$ and $\Gamma_{\nu\lambda}^\mu$ are the Christoffel symbols of the second kind for the Levi-Civita connection defined by G . Note that several terms such as π_A^i, Γ_{0i}^0 , and Γ_{00}^k will vanish in (8.7). From this we may write the differential equations for the integral curve of X_g^0 through a point $u \in L_V Y$. We obtain a geodesic equation,

$$\ddot{y}^A + \Gamma_{BC}^A \dot{y}^B \dot{y}^C + 2\Gamma_{B0}^A \dot{y}^B \dot{x}^0 + \Gamma_{00}^A \dot{x}^0 \dot{x}^0 = 0,$$

and an equation of parallel transport of vertical frames,

$$\dot{\pi}_B^A - \Gamma_{BD}^C \dot{y}^D \pi_C^A - \Gamma_{BD}^0 \dot{y}^D \pi_0^A - \Gamma_{B0}^C \dot{x}^0 \pi_C^A - \Gamma_{B0}^0 \dot{x}^0 \pi_0^A = 0.$$

This improves upon a “parallel transport” result of Norris [16, 17]. Indeed, Norris’s example is done on the linear frame bundle LQ and the symmetry group is just $O_g(k)$. In order for the Hamiltonian vector field to drive the parallel transport of a frame along a geodesic (that is, a solution to $\ddot{y}^A + \Gamma_{BC}^A \dot{y}^B \dot{y}^C = 0$), one must assume that one of the legs of the frame is tangent to the geodesic, restricting the types of frames that may be parallelly transported. However, in the present example, the choice of a nontrivial connection γ produces a parallel transport law for an *arbitrary* frame along a geodesic.

9 Conclusions

This investigation shows that the $(n+k)$ -symplectic geometry of the bundle of vertically adapted linear frames $L_V Y$ of a field configuration bundle Y extends the n -symplectic geometry of Norris [15, 16, 17] to provide momentum mappings for field theories. The “allowable” classical field momentum observables are a special case of these momentum mappings, generated by lifting a bundle automorphism of Y to create a momentum mapping on $L_V Y$. These momentum observables improve upon the momentum observables found in the multisymplectic geometry of Gotay, et al. [5, 6]. Examples of $(n+k)$ -symplectic momentum mappings produce: conservation of field angular momentum in the parameter space and in the total field theory, adjustments in angular momentum conservation laws to accommodate a change in inertial reference frame of a mechanical system, and generation of a parallel transport law for arbitrary frames along geodesics in a Riemannian manifold.

This work sets the stage for a theory of $(n+k)$ -symplectic reduction by symmetry. Reduction by symmetry in this context would address field theories with symmetry but would employ a finite-dimensional approach. Progress on the Lagrangian side of this problem is seen using the multisymplectic approach [2, 11] and using the n -symplectic approach [12, 18]. It is hoped that a full program of field-theoretic reduction by symmetry, analogous to cotangent bundle, Lie-Poisson, or Euler-Poincaré reduction, will emerge in the recent future.

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